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Stabilization of Euler–Bernoulli plate equation with variable coefficients by nonlinear boundary feedback

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Abstract

The aim of this paper is to investigate the uniform stabilization of Euler–Bernoulli plate equation with variable coefficients in the principle part subject to nonlinear boundary feedback laws. The exponential or rational energy decay rate is obtained by the multiplier method and the Riemannian geometry method.

Keywords: Euler–Bernoulli plate equation; Nonlinear boundary feedback; Exponential energy decay rate; Riemannian geometry method

1. Introduction

We consider the stabilization problem of an Euler–Bernoulli plate equation with variable coefficients and nonlinear boundary feedback. Where, for convenience, our problem starts out on a Riemannian manifold M of dimension 2 with a metric $g = \langle \cdot, \cdot \rangle$. For the classical case where

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$M = R^2$ and g is the dot product, the uniform stabilization of the Euler–Bernoulli plate by non-linear boundary feedback has been well studied by Rao [1], and Lasiecka and Triggiani [22,23].

The earlier attempt on the variable problems was made by Lagnese [2] through the classical analysis for the decay of the wave equation where the author imposed strict assumptions on the domain, see also Eller et al. [20] for variable coefficients thermoelastic. In our paper, we shall use the Riemannian geometry method to produce geometric multiplier in place of the classical ones so that by using Bochner technique in Riemannian geometry, some geometric multiplier identities (inequalities) are obtained, from which the stabilization results for the Euler–Bernoulli plate with variable coefficients are derived. This approach is first introduced into the boundary control problem by Yao [3] for the exact controllability of the wave equation where the principal operator has variable coefficients without the lower order term (the energy level). Then the approach is extended by Lasiecka et al. [4] to handle the first order term (the energy level) by using different multipliers where [3] was a preprint. As to the geometric approach, a different idea is used by Gulliver and Littman [5] to put their analysis of control problems on geodesics of the domain to produce a controllability condition described by geodesics.

Later the Riemannian geometry method is also applied to the observability inequality for the Euler–Bernoulli plate by Yao [6] and by Lasiecka et al. [7] for Carleman estimates including the lower order terms (the energy level). A thermoelastic plate is considered by Eller et al. [8] where the principal part is kept as the constant coefficients but with variable coupling terms. Recently, the thermoelastic plate with the variable principal part has been studied by Chai and Guo [9]. For a comparison of different methods on the variable problems, we refer to [4].

We would meet difficulty if we tried to study the variable coefficient problems by the same methods as used in the constant coefficient cases. This is because what we study, like exact controllability, feedback stabilization, etc., is a property of the metric which is controlled by the curvature of the metric. The constant coefficients case corresponds to the dot metric in R^n and, after we introduce an appropriate metric [3], the variable coefficients problems become a counterpart in the metric. Fortunately, the Bochner technique, developed in geometry [10], helps us a lot to overcome the complex of computation when we apply the multipliers to the problems.

In addition to handling the variable coefficient problem, we also obtain the stabilization results under weaker geometric assumption on the control portion of the boundary, thanks to some trace estimates of [11]. Such estimates were first established for the wave equation by Lasiecka and Triggiani [12,21].

It should be mentioned that the precise regularity possessed by solutions of the plate equation will be difficult to determine in general. However, when the boundary $\partial\Omega = \Gamma$ of the domain Ω is smooth enough, under some suitable assumptions on feedback functions, by the nonlinear semigroup theory and the elliptic regularity theory, we can characterize the regularity of the solution of plate equation. In fact, we can use the method—developed by Lasiecka and Tataru [24] for wave equation and used later by other authors in the context of other models—to prove the existence of smooth solutions of our problems (see Remark 3.2 for details).

Our paper is organized as follows. In Section 2, we introduce some notations with which we are working. In order to prove our main results, in Section 3, we establish some geometry identities (inequalities). The proofs of our main theorems are given in Section 4.

2. Some notations

We introduce some notations in Riemannian manifold in preparation for our systems of the Euler–Bernoulli plate with variable coefficients. It should be mentioned that all definitions and notations in this section are standard and classical in the literature. We refer the readers to [3] which is extremely useful in understanding much of the notations.

Let (M, g) be a Riemannian manifold with Riemannian metric $g = \langle\cdot,\cdot\rangle$. For each $x \in M$, M_x is the tangential space of M at x. We use $\chi(M)$ to denote the set of all vector fields on M. Denote the set of all n order tensor fields and the set of all n forms on M by $T^n(M)$ and $\Lambda^n(M)$, respectively, where n is a nonnegative integer.

It is well known that for each $x \in M$, the n order tensor space $T_x^n(M)$ on M_x is an inner product space, and its inner product $\langle\cdot,\cdot\rangle$ is defined in the following way: Let e_1, e_2 be a normal orthogonal basis of M_x, for any $\alpha, \beta \in T_x^n(M)$, $x \in M$, we define

$$\langle\alpha, \beta\rangle_{T_x^n} = \sum_{i_1, i_2, \ldots, i_n = 1}^{2} \alpha(e_{i_1}, \ldots, e_{i_n})\beta(e_{i_1}, \ldots, e_{i_n}), \quad \text{at } x. \tag{2.1}$$

Let Ω be a bounded domain of M with smooth boundary Γ, then $T^n(\Omega)$ is an inner product space with inner product $(\cdot,\cdot)_{T^n(\Omega)}$ in the following sense:

$$(T_1, T_2)_{T^n(\Omega)} = \int_\Omega \langle T_1, T_2\rangle_{T_x^n}\, dx, \quad T_1, T_2 \in T^n(\Omega), \tag{2.2}$$

where dx is the volume element of M in its Riemannian metric g.

The completion of $T^n(\Omega)$ in the inner product (2.2) is denoted by $L^2(\Omega, T^n)$, in particular, we have $L^2(\Omega, \Lambda) = L^2(\Omega, T)$. The completion of $C^\infty(\Omega)$ in the following inner product (2.3) is denoted by $L^2(\Omega)$:

$$(f, h)_{L^2(\Omega)} = \int_\Omega f(x)h(x)\, dx, \quad f, h \in C^\infty(\Omega). \tag{2.3}$$

Let D be the Levi–Civita connection on M in the Riemannian metric g. For $U \in \chi(M)$, DU is the covariant differential of U which is a second order covariant tensor field in the following sense:

$$DU(X, Y) = D_Y U(X) = \langle D_Y U, X\rangle, \quad \forall X, Y \in M_x,\ x \in M. \tag{2.4}$$

For any $T \in T^2(M)$, the trace of T at x is defined by

$$\operatorname{tr} T = \sum_{i=1}^{2} T(e_i, e_i), \tag{2.5}$$

where e_1, e_2 is an orthonormal basis of M_x. It is obviously that $\operatorname{tr} T \in C^\infty(M)$ if $T \in T^2(M)$. The exterior derivative $d : \Lambda^n(M) \to \Lambda^{n+1}(M)$ satisfies $d^2 = 0$. For each d, there is a first order differential operator $\delta : \Lambda^{n+1}(M) \to \Lambda^n(M)$, which is the formal adjoint operator of d and is characterized by

$$(d\alpha, \beta)_{L^2(\Omega, \Lambda^{n+1})} = (\alpha, \delta\beta)_{L^2(\Omega, \Lambda^n)},$$

for $\alpha \in \Lambda^n(\Omega)$ and $\beta \in \Lambda^{n+1}(\Omega)$ with compact support.

The Sobolev space $H^n(\Omega)$ is the completion of $C^\infty(\Omega)$ with respect to the norm

$$\|f\|^2_{H^n(\Omega)} = \sum_{i=1}^{n} \|D^i f\|^2_{L^2(\Omega, T^n)} + \|f\|^2_{L^2(\Omega)}, \quad \text{for } f \in C^\infty(\Omega),$$

where $D^i f$ is the ith covariant differential of f in metric g, and $\|\cdot\|_{L^2(\Omega,T^n)}$, $\|\cdot\|_{L^2(\Omega)}$ are the induced norms in inner products (2.2), (2.3), respectively. For details on Sobolev spaces on Riemannian manifolds, we refer to [14] or [15].

The following Green formulae are due to [15, Chapter 2, §10]:

$$(d\alpha, \beta)_{L^2(\Omega,\Lambda^{n+1})} = (\alpha, \delta\beta)_{L^2(\Omega,\Lambda^n)} + \int_\Gamma \langle \nu \wedge \alpha, \beta\rangle_{T_x^{n+1}}\, d\Gamma, \tag{2.6}$$

for $\alpha \in \Lambda^n(\bar{\Omega})$ and $\beta \in \Lambda^{n+1}(\bar{\Omega})$,

$$(\delta\alpha, \beta)_{L^2(\Omega,\Lambda^n)} = (\alpha, d\beta)_{L^2(\Omega,\Lambda^{n+1})} - \int_\Gamma \langle l_\nu \wedge \alpha, \beta\rangle_{T_x^n}\, d\Gamma, \tag{2.7}$$

for $\alpha \in \Lambda^{n+1}(\bar{\Omega})$ and $\beta \in \Lambda^n(\bar{\Omega})$. Where $d\Gamma$ is the line element of Γ, ν is the unit normal of Γ pointing towards the exterior of Γ. For $\alpha \in \Lambda^{n+1}(\bar{\Omega})$ and ν, $l_\nu\alpha \in T^n(\bar{\Omega})$ is defined by

$$l_\nu\alpha(X_1, \dots, X_n) = \alpha(\nu, X_1, \dots, X_n), \quad \forall X_1, \dots, X_n \in \chi(\bar{\Omega}),$$

where $\wedge$ is the exterior product of differential forms.

In the case of dimension 2, Ricci tensor is a second order covariant tensor field which is given by

$$\mathrm{Ricci}(X, Y)(x) = \sum_{i=1}^{2} R(e_i, X, e_i, Y), \quad \forall X, Y \in M_x,\ x \in M, \tag{2.8}$$

where e_1, e_2 is an orthonormal basis of M_x, and R is the curvature tensor of the Levi–Civita connection D (for details see [16]). It is easy to check from (2.8) that

$$\mathrm{Ricci}(X, Y) = k(x)\langle X, Y\rangle, \quad \forall X, Y \in M_x,\ x \in M, \tag{2.9}$$

where $k(x)$ is the Gaussian curvature function on M. We denote by $\Delta : C^2(R^2) \to C^2(R^2)$ the Laplace operator in the Riemannian metric g. Then we have

$$\Delta h = \frac{1}{\sqrt{G(x)}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{G(x)}\, g_{ij}^{-1}(x) \frac{\partial h}{\partial x_j}\right), \quad \forall h \in C^2\big(R^2\big), \tag{2.10}$$

where $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, $G(x) = \det(g_{ij})$, and $g_{il}g^{lj} = \delta_i^j$, $x = (x_1, x_2)$ is the classical coordinate system.

It follows from [3, Lemma 2.1] that

$$\Delta h = \sum_{i,j=1}^{n} g_{ij}^{-1} D^2 h\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \quad \forall h \in C^2\big(R^2\big). \tag{2.11}$$

We will use many times the following divergence formulae:

$$\int_\Omega \operatorname{div} X\, dx = \int_\Gamma \langle X, \nu\rangle\, d\Gamma, \tag{2.12}$$

where $\operatorname{div} X$ is the divergence of vector field X in Riemannian metric g, ν is the normal of Γ pointing towards the exterior of Γ.

3. Geometry inequality and the main results

We keep all the notations as in Section 2. Let Ω be a bounded domain in the Riemannian manifold (M, g) with smooth boundary Γ. Assume that Ω is occupied by the middle surface of the plate in equilibrium. Let $y(x,t)$ be the vertical displacement of the point $x \in \Omega$ at time t, assume that the material under going change obeys Hooke's Law. We will establish the exponential or rational energy decay rate for the solution of the following system:

$$\begin{cases} y'' + \Delta^2 y - (1-\mu)\delta(kdy) = 0, & \text{in } \Omega \times [0,\infty), \\ \Delta y + (1-\mu)B_1 y = v_1 & \text{on } \Gamma \times [0,\infty), \\ \partial_\nu \Delta y + (1-\mu)B_2 y = v_2 & \text{on } \Gamma \times [0,\infty), \\ y(0) = y_0, \quad y'(0) = y_1 & \text{in } \Omega, \end{cases} \tag{3.1}$$

where ν is the unit normal vector along Γ pointing towards the exterior of Γ, $\Delta : C^2(M) \to C^2(M)$ is the Laplace operator in the Riemannian metric g. In the above equations, k is the Gaussian curvature function on Ω , $0 < \mu < \frac{1}{2}$ is the Poisson coefficient, d is the exterior derivative, δ is the formal adjoint operator of d, B_1, B_2 are the boundary operators defined by

$$B_1 y = -D^2 y(\tau, \tau), \tag{3.2}$$

$$B_2 y = \frac{\partial}{\partial \tau}\big(D^2 y(\tau, \nu)\big) + k\partial_\nu y, \tag{3.3}$$

where $D^2 y$ is the Hessian of y, it is a second order tensor. τ is the unit tangential vector along the boundary Γ, $\partial_\nu y = \frac{\partial y}{\partial \nu} = \langle \nu, Dy\rangle$ is the normal derivative.

Remark 3.1. The term $(1-\mu)\delta(kdy)$ in the system (3.1) comes from the curvedness of the Riemannian metric g. For the flat case where $M = R^2$ and $k = 0$, system (3.1) is the same as in [1].

Remark 3.2. In general, the precise regularity of the solution of the plate equation will not be easy to determine. However, since the boundary of the domain is smooth enough, we can obtained the regularity of the solution of the Euler–Bernoulli plate equation we need in this paper by the same method developed by Lasiecka and Tataru [24] for wave equation. More precisely, as Lasiecka and Tataru [24], we first consider an approximation scheme that allows us to obtain the uniform estimates on the approximation solutions. Then we show that the solutions of the approximated problem converge to solutions of the original problem. Finally, we prove the result for the original problem by passing the limit, we refer the reader for [24] for the details of the proof. As a result, under the assumptions that g, h are continuous and monotone and satisfy some suitable growth conditions (see (3.17)–(3.20)), we can prove for any initial data $(y_0, y_1) \in D$, the system (3.1) admits a unique strong solution such that

$$(y, y_t) \in D, \quad y \in W^{1,\infty}\big(R^+, H^2(\Omega)\big), \quad \Delta^2 y \in L^\infty\big(R^+, L^2(\Omega)\big),$$

where

$$D = \left\{ (y,z) \,\middle|\, y \in H^2(\Omega),\ z \in H^2(\Omega),\ \Delta^2 y \in L^2(\Omega),\right.$$
$$\Delta y + (1-\mu)B_1 y = -\beta \frac{\partial y}{\partial \nu} - h\left(\frac{\partial z}{\partial \nu}\right) \in L^2(\Gamma),$$
$$\left.\frac{\partial}{\partial \nu}\Delta y + (1-\mu)B_2 y = \alpha y + g(z) \in L^2(\Gamma)\right\}.$$

Here, the traces are defined in the sense of distributions. However, by Green's formula, we see that the traces defined in this way coincide with the usual ones when both two definitions of the traces make sense.

If the initial data $(y_0, y_1) \in V = H^2(\Omega) \times L^2(\Omega)$, then the system (3.1) admits a unique weak solution with the regularity

$$(y, y_t) \in V, \quad y \in C^0\big(R^+, H^2(\Omega)\big) \cap C^1\big(R^+, L^2(\Omega)\big).$$

In the present work, we will use the following nonlinear boundary feedback laws:

$$v_1 = -\beta \frac{\partial y}{\partial \nu} - h\left(\frac{\partial y'}{\partial \nu}\right), \qquad v_2 = \alpha y + g(y'). \tag{3.4}$$

For a solution $y(x,t)$ of the system (3.1) with the feedback laws (3.4), we define the associated energy by

$$E(t) = \frac{1}{2}\left\{ \int_\Omega \big[|y'|^2 + a(y,y)\big]\,dx + \int_\Gamma \left(\alpha|y|^2 + \beta\left|\frac{\partial y}{\partial \nu}\right|^2\right) d\Gamma \right\}, \tag{3.5}$$

where for any (y,u), we have set

$$a(y,u) = (1-\mu)\big\langle D^2 y, D^2 u\big\rangle_{T_x^2} + \mu\big(\operatorname{tr} D^2 y \operatorname{tr} D^2 u\big). \tag{3.6}$$

The following formula is a key to our problem, which is something like classical Green's formula presenting the relationship between the interior and the boundary.

Lemma 3.1. *Let $y, u \in H^4(\Omega)$ be given such that all the terms in the following formulae are well defined. Assume that Γ is a closed curve. Then we have*

$$\int_\Omega \big[\Delta^2 y - (1-\mu)\delta(k\,dy)\big] u\,dx = \int_\Omega a(y,u)\,dx - \int_\Gamma \big[\Delta y + (1-\mu)B_1 y\big] \frac{\partial u}{\partial \nu}\,d\Gamma + \int_\Gamma \left[\frac{\partial(\Delta y)}{\partial \nu} + (1-\mu)B_2 y\right] u\,d\Gamma. \tag{3.7}$$

Proof. Since y is a function, we have $\delta y = 0$, and

$$\delta \Delta_H\,dy = \delta\,d\delta\,dy = \Delta_H^2 y, \tag{3.8}$$

where Δ_H is the Hodge–Laplacian operator on forms, and $\Delta_H = -\Delta$, $\Delta_H d = -d\Delta$, when they are applied to function y.

Since $dy = Dy$, it follows from [3, Theorem 2.2], (2.6) and (3.8) that

$$\int_\Omega \langle D^2 y, D^2 u\rangle_{T_x^2}\,dx$$
$$= (D\,dy, D\,du)_{L^2(\Omega,T^2)} = \int_\Omega \langle D\,dy, D\,du\rangle\,dx$$
$$= \int_\Omega \big[(\Delta_H\,dy - k\,dy)\,du\big]\,dx + \int_\Gamma \langle D_\nu\,dy, du\rangle\,d\Gamma$$
$$= \int_\Omega \big(\Delta_H^2 y - \delta(k\,dy)\big)u\,dx$$
$$+ \int_\Gamma u\left[\langle \nu, \Delta_H\,dy\rangle - k\frac{\partial y}{\partial \nu}\right]d\Gamma + \int_\Gamma D^2 y(\nu, du)\,d\Gamma$$
$$= \int_\Omega \big[\big(\Delta^2 y - \delta(k\,dy)\big)u\big]\,dx$$
$$+ \int_\Gamma \left[D^2 y(\nu,\nu)\frac{\partial u}{\partial \nu} + D^2 y(\nu,\tau)\frac{\partial u}{\partial \tau}\right]d\Gamma - \int_\Gamma u\left[\frac{\partial \Delta y}{\partial \nu} + k\frac{\partial y}{\partial \nu}\right]d\Gamma. \tag{3.9}$$

Since $\operatorname{tr} D^2 y = \Delta y$, by (2.6), we have

$$\int_\Omega \operatorname{tr} D^2 y \operatorname{tr} D^2 u\,dx = \int_\Omega \Delta^2 y u\,dx + \int_\Gamma \left[\Delta y\frac{\partial u}{\partial \nu} - u\frac{\partial \Delta y}{\partial \nu}\right]d\Gamma. \tag{3.10}$$

Since Γ is a closed curve,

$$\int_\Gamma D^2 y(\nu,\tau)\frac{\partial u}{\partial \tau}\,d\Gamma = -\int_\Gamma u\frac{\partial}{\partial \tau}\big(D^2 y(\nu,\tau)\big)\,d\Gamma. \tag{3.11}$$

Furthermore, we have

$$\Delta y = D^2 y(\nu,\nu) + D^2 y(\tau,\tau), \quad \text{on } \Gamma. \tag{3.12}$$

By (3.9)–(3.12) and (3.6), we get (3.7), and Lemma 3.1 is proved. □

By using the formulae (3.7), we have

$$\frac{d}{dt}E(t) = -\int_\Gamma \left\{g(y')y' + h\left(\frac{\partial y'}{\partial \nu}\right)\frac{\partial y'}{\partial \nu}\right\}d\Gamma. \tag{3.13}$$

In addition, if we assume that g and h are nondecreasing continuous functions such that $g(0) = h(0) = 0$, then it follows from (3.13) that the system (3.1) with the feedback laws (3.4) is dissipative in the sense that the associated energy $E(t)$ is nonincreasing.

Let H be a vector field on Riemannian manifold (M, g) such that

$$DH(X,X) = b(x)|X|^2, \quad \forall X \in M_x,\ x \in \bar{\Omega}, \tag{3.14}$$

where $b(x)$ is a function on Ω. We also assume that the vector H satisfies $\min_{x\in\Omega} b(x) \geqslant 1$ such that

$$H.\nu \geqslant 0, \quad \forall x \in \Gamma. \tag{3.15}$$

Remark 3.3. Geometric condition (3.14) is used in [6] for some observability inequalities of the Euler–Bernoulli equation with variable coefficients. In our paper, this condition is needed for the computation of the geometrical multiplier method. We remark that this condition has nothing to do with the shape of the domain Ω. For any Riemannian manifold M, the existence of such a vector field on $\Omega \subset M$ has been proved in Yao [17]. For the classical Euclidean metric, we take $H = x - x_0$ and $DH(x,x) = |x|^2$, the above assumption is true with $b(x) = 1$. One can also find some other nontrivial examples in [6]. This condition is also used in Lasiecka and Triggiani [19, (1.23)] for uniform stabilization of a shallow shell model with nonlinear boundary feedbacks.

Here the geometric condition (3.15) is generally considered to be much weaker than the following

$$H.\nu > 0, \quad \forall x \in \Gamma, \tag{3.16}$$

which is used in the literature to avoid the complex boundary trace estimates.

Our main results are:

Theorem 3.1. *Assume*

$$\alpha(x) \in L^\infty(\Gamma), \quad \alpha_1 \geqslant \alpha(x) \geqslant \alpha_0 > 0, \quad \forall x \in \Gamma, \tag{3.17}$$

$$\beta(x) \in L^\infty(\Gamma), \quad \beta_1 \geqslant \beta(x) \geqslant \beta_0 > 0, \quad \forall x \in \Gamma, \tag{3.18}$$

$$g(x) \in C^0(R), \quad \textit{nondecreasing}, \quad g(0) = 0, \quad g(s)s > 0, \quad \forall s \neq 0, \tag{3.19}$$

$$h(x) \in C^0(R), \quad \textit{nondecreasing}, \quad h(0) = 0, \quad h(s)s > 0, \quad \forall s \neq 0, \tag{3.20}$$

then for every solution y of system (3.1) *with feedback laws* (3.4)*, we have*

(i) *If there exist $L_0, L_1, L_2, L_3 > 0$ such that*

$$\frac{|s|}{L_1} \leqslant \left|g(s)\right| \leqslant L_0|s|, \quad \frac{|s|}{L_3} \leqslant \left|h(s)\right| \leqslant L_2|s|, \quad \forall s \in R, \tag{3.21}$$

then for any given $M > 1$, there exists a constant $\omega > 0$ such that

$$E(t) \leqslant ME(0)\exp(-\omega t), \quad \forall t \geqslant 0. \tag{3.22}$$

(ii) *If there exist $L_0, L_1, L_2, L_3 > 0$ and $p > 1$ such that*

$$\frac{1}{L_1}\min\left\{|s|, |s|^p\right\} \leqslant \left|g(s)\right| \leqslant L_0|s|, \quad \forall s \in R, \tag{3.23}$$

$$\frac{1}{L_3}\min\left\{|s|, |s|^p\right\} \leqslant \left|h(s)\right| \leqslant L_2|s|, \quad \forall s \in R, \tag{3.24}$$

then for any given $M > 1$, there exists a constant $\omega > 0$, depending continuously on $E(0)$ such that

$$E(t) \leqslant ME(0)(1+\omega t)^{\frac{-2}{p-1}}, \quad \forall t \geqslant 0. \tag{3.25}$$

Theorem 3.2. *Assume* (3.17)–(3.20). *In addition, if there exist* $L_0, L_1, L_2, L_3 > 0$ *and* $p < 1$ *such that*

$$\frac{|s|}{L_1} \leqslant |g(s)| \leqslant L_0^p \max\{|s|, |s|^p\}, \quad \forall s \in R, \tag{3.26}$$

$$\frac{|s|}{L_3} \leqslant |h(s)| \leqslant L_2^p \max\{|s|, |s|^p\}, \quad \forall s \in R, \tag{3.27}$$

then for any given $M > 1$, *there exists a constant* $\omega > 0$, *depending continuously on* $E(0)$, *such that*

$$E(t) \leqslant ME(0)(1 + \omega t)^{\frac{-2p}{1-p}}, \quad \forall t \geqslant 0, \tag{3.28}$$

for every solution y *of system* (3.1) *with feedback laws* (3.4).

Remark 3.4. We see that the estimate (3.25) is similar to (3.28) provided p is replaced by $\frac{1}{p}$. The same estimate has been obtained for the case of constant coefficients (see [1]) but with stronger conditions (3.16). In case of variable coefficients, if we assume, instead of (3.15), (3.16) is true, then we can get the exact constant ω in (3.25) and (3.28).

Remark 3.5. By a standard arguments as in the case of constant coefficients, and because of the density, it is sufficient to consider the smooth solutions.

Lemma 3.2. *Let* H *satisfies* (3.14). *We have*

$$\int_\Omega a(y, H(y))\, dx = \frac{1}{2} \int_\Gamma a(y, y)\langle H, \nu\rangle\, d\Gamma + \int_\Omega ba(y, y)\, dx + \operatorname{lot}(y), \tag{3.29}$$

where $\operatorname{lot}(y)$ *denotes the lower order term with respect to the energy* (3.5).

Proof. Given $x \in \Omega$. Let E_1, E_2 be a frame field normal at x. By the following identity (see [16, §2, Lemma 4]):

$$D^2T(\ldots, X, Y) = D^2T(\ldots, Y, X) + (R_{XY}T)(\ldots), \tag{3.30}$$

we have

$$\begin{aligned} D^2(H(y))(E_i, E_j) &= E_jE_i(Dy(H)) = E_j(D^2y(E_i, H) + Dy(D_{E_i}H)) \\ &= D^3y(E_i, H, E_j) + D^2y(E_i, D_{E_j}H) + E_j\langle Dy, D_{E_i}H\rangle \\ &= D_H(D^2y)(E_i, E_j) + R(Dy, E_i, H, E_j) \\ &\quad + D^2y(E_i, D_{E_j}H) + E_j\langle Dy, D_{E_i}H\rangle, \quad \text{at } x. \end{aligned} \tag{3.31}$$

Since $(D_{E_i}E_j)(x) = 0$ for $1 \leqslant i, j \leqslant 2$,

$$\begin{aligned} E_j\langle Dy, D_{E_i}H\rangle &= D^2H(Dy, E_i, E_j) + DH(D_{E_j}Dy, E_i) \\ &= l_{Dy}D^2H(E_i, E_j) + D^2y(D_{E_i}H, E_j), \quad \text{at } x. \end{aligned} \tag{3.32}$$

Inserting (3.32) into (3.31) yields

$$D^2(H(y)) = D_H(D^2y) + D^2y(\cdot, D.H) + D^2y(D.H, \cdot) - l(y), \tag{3.33}$$

where $l(y) = -R(Dy, \cdot, H, \cdot) - D^2H(Dy, \cdot, \cdot)$, and "$\cdot$" denotes the position of the variable.

On the other hand, for given $x \in \Omega$, let e_1, e_2 be a frame field normal at x. By direct computation, we have

$$H\big(\operatorname{tr} D^2 y\big) = \sum_{i=1}^{2} D_H D^2 y(E_i, E_i) = \operatorname{tr}\big(D_H D^2 y\big), \quad \text{at } x. \tag{3.34}$$

Since $D^2 y$ is a symmetric second order tensor field, it follows from [17, Lemma 2.7] and (3.30) that

$$\big\langle D^2 y, D^2\big(H(y)\big)\big\rangle_{T_x^2} = \frac{1}{2} H\big(\big|D^2 y\big|_{T_x^2}^2\big) + 2b\big|D^2 y\big|^2 + \big\langle D^2 y, l(y)\big\rangle_{T_x^2}, \tag{3.35}$$

$$\operatorname{tr} D^2 y \operatorname{tr} D^2\big(H(y)\big) = \frac{1}{2} H\big(\big(\operatorname{tr} D^2 y\big)^2\big) + 2b\big(\operatorname{tr} D^2 y\big)^2 + \operatorname{tr} D^2 l(y). \tag{3.36}$$

Combining the divergence formula with (3.35) and (3.36), we obtain (3.29). The proof is completed. □

Lemma 3.3. *Let $w \in H^4(\Omega)$ be such that*

$$\begin{cases} \Delta^2 w \in L^2(\Omega), \\ \Delta w + (1-\mu) B_1 w = v_1 \in L^2(\Gamma), \\ \partial_\nu \Delta w + (1-\mu) B_2 w = v_2 \in L^2(\Gamma). \end{cases} \tag{3.37}$$

Then we have

$$\begin{aligned} &-\int_\Omega \big[\Delta^2 w - (1-\mu)\delta(k\,dw)\big] H(w)\,dx \\ &\quad \leqslant -\frac{1}{2}\int_\Omega a(w,w)\,dx + \operatorname{lot}(w) + C\int_\Gamma \left(|v_1|^2 + |v_2|^2 + \alpha|w|^2 + \beta\left|\frac{\partial w}{\partial \nu}\right|^2\right) d\Gamma, \end{aligned} \tag{3.38}$$

where $C > 0$ is a constant independent of function w, and $\operatorname{lot}(w)$ *is the lower term with respect to the energy.*

Proof. For simple reason, we start with $v_1 \in H^{\frac{3}{2}}(\Gamma)$ and $v_2 \in H^{\frac{1}{2}}(\Gamma)$. First, we apply Theorem 2.2 of [11] to obtain an estimate of $D^2 w$ on the boundary which will be crucial to avoid the geometric condition (3.16). We take α in [11] as α_0 and let $1/2 > \alpha_0 > 0$, $1/2 > \epsilon > 0$, and $1/2 > s_0 > 0$ be fixed. In particular, we take T of [11] to be 1 and $\Gamma_1 = \Gamma$. Since the function w here is independent of time t ($w_t = w_{tt} = 0$), Theorem 2.2 of [11] implies the following inequality:

$$\begin{aligned} (1-2\alpha_0)\int_\Gamma \big|D^2 w\big|^2\,d\Gamma &= \big\|D^2 w\big\|^2_{L^2([\alpha_0, 1-\alpha_0];\, L^2(\Gamma, T^2))} \\ &\leqslant C_{\alpha_0,\epsilon,1}\Big\{\big\|\Delta^2 w\big\|^2_{H^{-s_0}(Q^1)} + \|v_1\|^2_{L^2(\Sigma^1)} \\ &\quad + \|v_2\|^2_{H^{-1}(\Sigma^1)} + \|w\|^2_{L^2([0,1];\, H^{3/2+\epsilon}(\Omega))}\Big\}, \end{aligned} \tag{3.39}$$

where $C_{\alpha_0,\epsilon,1} > 0$ is a constant independent of w, and

$$Q^1 = (0,1) \times \Omega, \qquad \Sigma^1 = (0,1) \times \Gamma.$$

It is clear that there is $C_1 > 0$ such that

$$\|\Delta^2 w\|^2_{H^{-s_0}(Q^1)} \leqslant C_1 \|\Delta^2 w\|^2_{L^2(Q^1)} = C_1 \|\Delta^2 w\|^2_{L^2(\Omega)}$$

and

$$\|v_2\|^2_{H^{-1}(\Sigma^1)} \leqslant C_1 \|v_2\|^2_{L^2(\Sigma^1)} = C_1 \|v_2\|^2_{L^2(\Gamma)}.$$

Thus, inequality (3.39) means that there is $C_2 > 0$ such that

$$\int_\Gamma |D^2 w|^2 \, d\Gamma \leqslant C_2 \big(\|\Delta^2 w\|^2_{L^2(\Omega)} + \|v_1\|^2_{L^2(\Gamma)} + \|v_2\|^2_{L^2(\Gamma)}\big) + \mathrm{lot}(w), \tag{3.40}$$

for any $w \in H^4(\Omega)$.

Since $w \in H^4(\Omega)$, by Lemmas 3.1 and 3.2, we have

$$\begin{aligned}
&\int_\Omega \big[\Delta^2 w - (1-\mu)\delta(k\, dw)\big] H(w)\, dx \\
&\quad = \int_\Omega ba(w,w)\, dx + \int_\Gamma v_2\big(H(w)\big)\, d\Gamma \\
&\qquad - \int_\Gamma v_1 \partial_\nu \big(H(w)\big)\, d\Gamma + \frac{1}{2}\int_\Gamma a(w,w)\langle H, \nu\rangle\, d\Gamma + \mathrm{lot}(w).
\end{aligned} \tag{3.41}$$

Thus

$$\begin{aligned}
&\int_\Omega \big[\Delta^2 w - (1-\mu)\delta(k\, dw)\big]\big(H(w)\big)\, dx \\
&\quad \geqslant \int_\Omega a(w,w)\, dx + \int_\Gamma v_2\big(H(w)\big)\, d\Gamma - \int_\Gamma v_1 \partial_\nu\big(H(w)\big)\, d\Gamma + \mathrm{lot}(w).
\end{aligned} \tag{3.42}$$

Now a straightforward computation shows that there is $C_3 > 0$ independent of w such that

$$\big|\partial_\nu\big(H(w)\big)\big| \leqslant C_3\big(|\partial_\nu w| + |D^2 w|_{T_x^2}\big). \tag{3.43}$$

In addition, from the inequality (3.40), we have

$$\begin{aligned}
&C_3 \int_\Gamma \big|v_1 \partial_\nu\big(H(w)\big)\big|\, d\Gamma \\
&\quad \leqslant \frac{\mu}{2C_2}\int_\Gamma |D^2 w|^2\, d\Gamma + \frac{C_3^2 C_2}{2\mu}\int_\Gamma |v_1|^2\, d\Gamma \\
&\quad \leqslant \frac{\mu}{2}\int_\Omega \big(\Delta^2 w\big)^2\, dx + C_4 \int_\Gamma \big(|v_1|^2 + |v_2|^2\big)\, \Gamma.
\end{aligned} \tag{3.44}$$

Next, we estimate the term $\int_\Gamma v_2(H(w))\, d\Gamma$. Let $\sigma > 0$ be such that

$$\int_\Gamma |\nabla w|^2\,d\Gamma \leqslant \sigma\left\{\int_\Omega \left|D^2 w\right|_{T_x^2}\,dx + \int_\Gamma \left(\alpha|w|^2 + \beta\left|\frac{\partial w}{\partial \nu}\right|^2\right)d\Gamma\right\}, \tag{3.45}$$

for any $w \in H^4(\Omega)$. It follows from the inequality (3.45) that

$$\begin{aligned}
&\int_\Gamma |v_2|\left|H(w)\right|d\Gamma \\
&\quad\leqslant \sigma_1 \int_\Gamma |v_2||\nabla w|\,d\Gamma \\
&\quad\leqslant \frac{1-\mu}{2\sigma}\int_\Gamma |\nabla w|^2\,d\Gamma + \frac{\sigma_1^2\sigma}{2(1-\mu)}\int_\Gamma |v_2|^2\,d\Gamma \\
&\quad\leqslant \frac{1-\mu}{2}\int_\Omega \left|D^2 w\right|_{T_x^2}^2\,dx + C_5\int_\Gamma \left(|v_2|^2 + \alpha|w|^2 + \beta\left|\frac{\partial w}{\partial \nu}\right|^2\right)d\Gamma,
\end{aligned} \tag{3.46}$$

where $\sigma_1 = \sup_{x\in\Gamma}|H|$.

Note that $a(w,w) = (1-\mu)|D^2 w|_{T_x^2}^2 + \mu(\Delta^2 w)^2$, we obtain the inequality (3.38) after substituting the inequalities (3.43), (3.44), (3.46) into the right-hand side of the inequality (3.42).

For the general case when $v_1 \in L^2(\Gamma)$, $v_2 \in L^2(\Gamma)$, by the same density arguments as in [1], we complete the proof of Lemma 3.3. □

Lemma 3.4. *For any given $y \in H^4(\Omega)$, the solution u of the following problem*

$$\begin{cases} \Delta^2 u - (1-\mu)\delta(k\,du) = 0 & \text{in } \Omega, \\ u = y & \text{on } \Gamma, \\ \frac{\partial u}{\partial \nu} = \frac{\partial y}{\partial \nu} & \text{on } \Gamma, \end{cases} \tag{3.47}$$

satisfies the following estimates:

$$\int_\Omega u^2\,dx \leqslant \gamma^2 \int_\Gamma \left\{|y|^2 + \left|\frac{\partial y}{\partial \nu}\right|^2\right\}d\Gamma, \tag{3.48}$$

$$a(y,u) = a(u,u) \geqslant 0, \tag{3.49}$$

where γ is a constant depending only on the domain Ω.

Proof. (3.48) is a classical result of the elliptic theory.

Now we prove (3.49). We start with $y \in H^4(\Omega)$, then we deduce that $u \in H^4(\Omega)$. By Lemma 3.1, we have

$$\begin{aligned}
0 &= \int_\Omega \left[\Delta^2 u - (1-\mu)\delta(k\,du)\right](u-y)\,dx \\
&= \int_\Omega a(u, u-y) - \int_\Gamma \left[\Delta u + (1-\mu)B_1 u\right]\frac{\partial}{\partial \nu}(u-y)\,d\Gamma \\
&\quad + \int_\Gamma \left[\frac{\partial \Delta u}{\partial \nu} + (1-\mu)B_2 u\right](u-y)\,d\Gamma.
\end{aligned} \tag{3.50}$$

Using the boundary conditions in (3.47), it follows from (3.50) that

$$a(y,u)=a(u,u)\geqslant 0. \tag{3.51}$$

By a standard arguments of density, we complete the proof of Lemma 3.4. □

Now, we introduce the product space $V=H^2(\Omega)\times L^2(\Omega)$. Let

$$\begin{aligned} D=\biggl\{(y,z)\,\bigg|\, & y\in H^2(\Omega),\ z\in H^2(\Omega),\ \Delta^2 y\in L^2(\Omega),\\ & \Delta y+(1-\mu)B_1 y=-\beta\frac{\partial y}{\partial\nu}-h\left(\frac{\partial z}{\partial\nu}\right)\in L^2(\Gamma),\\ & \frac{\partial}{\partial\nu}\Delta y+(1-\mu)B_2 y=\alpha y+g(z)\in L^2(\Gamma)\biggr\}, \end{aligned} \tag{3.52}$$

where the traces in (3.52) are defined in the sense of distributions. However, by Green's formula, we see that the traces defined in this way coincide with the usual ones when both two definitions of the traces make sense.

Remark 3.6. Under our assumptions (3.17)–(3.20), if there exists a constant $C>0$ such that $|h(s)|\leqslant C(1+|s|)$, $\forall s\in R$, then the terms in the right-hand side of (3.52) belong to $L^2(\Gamma)$.

Let $(y_0,y_1)\in D$ and y be the regularity solution of the system (3.1). We introduce the functional

$$p(t)=\int_\Omega y'H(y)\,dx+C_0\int_\Omega y'u\,dx,$$

where u is the solution of Eq. (3.47) and C_0 is a positive constant to be determined later.

Lemma 3.5. *There exist some constants γ_0, C_1, C_2, C_3 such that*

$$\bigl|p(t)\bigr|\leqslant C_1E(t),\quad \forall t\geqslant 0, \tag{3.53}$$

$$p'(t)\leqslant -\gamma_0E(t)+C_2\int_\Gamma\left\{|y'|^2+\left|\frac{\partial y'}{\partial\nu}\right|^2\right\}d\Gamma+C_3\int_\Gamma\left\{g^2(y')+h^2\left(\frac{\partial y'}{\partial\nu}\right)\right\}d\Gamma. \tag{3.54}$$

Proof. By applying Lemma 3.4 and the Cauchy–Schwartz inequality to $p(t)$, the estimate (3.53) is easily verified. Indeed, the constant C_1 is given by $C_1=\gamma C_0(\frac{1}{\alpha_0}+\frac{1}{\beta_0})^{\frac12}+\lambda_0R$, where λ_0 is the constant such that

$$\|\nabla y\|^2_{H^1(\Omega)}\leqslant\lambda_0^2\left\{\int_\Omega a(y,y)\,dx+\int_\Gamma\left(\alpha|y|^2+\beta\left|\frac{\partial y}{\partial\nu}\right|^2\right)d\Gamma\right\},\quad \forall y\in H^2(\Omega). \tag{3.55}$$

Since

$$p'(t)=-\int_\Omega\bigl[\Delta^2 y-(1-\mu)\delta(k\,dy)\bigr]H(y)\,dx$$

$$- \frac{1}{2} \int_\Omega \operatorname{div} H |y'|^2 \, dx + \frac{1}{2} \int_\Gamma H.\nu |y'|^2 \, d\Gamma$$
$$- C_0 \int_\Omega \big[\Delta^2 y - (1-\mu)\delta(k\,dy)\big] u \, dx + C_0 \int_\Omega y' u' \, dx. \tag{3.56}$$

By Lemma 3.1, we have

$$\begin{aligned}
&-C_0 \int_\Omega \big[\Delta^2 y - (1-\mu)\delta(k\,dy)\big] u \, dx \\
&\quad = -C_0 \int_\Omega a(y,u)\, dx - C_0 \int_\Gamma \left\{ v_2 u - v_1 \frac{\partial u}{\partial \nu} \right\} d\Gamma \\
&\quad \leqslant -C_0 \int_\Omega a(u,u)\, dx - C_0 \int_\Gamma \left\{ v_2 y - v_1 \frac{\partial y}{\partial \nu} \right\} d\Gamma \\
&\quad \leqslant -C_0 \int_\Gamma \left\{ v_2 y - v_1 \frac{\partial y}{\partial \nu} \right\} d\Gamma.
\end{aligned} \tag{3.57}$$

Using Lemma 3.4 and Cauchy–Schwartz inequality, we have

$$C_0 \int_\Omega y' u' \, dx \leqslant \frac{1}{2} \int_\Omega |y'|^2 \, dx + \frac{(\gamma C_0)^2}{2} \int_\Gamma \left\{ |y'|^2 + \left| \frac{\partial y'}{\partial \nu} \right|^2 \right\} d\Gamma. \tag{3.58}$$

Inserting (3.57), (3.58) into (3.56), we obtain

$$\begin{aligned}
p'(t) \leqslant{} & - \int_\Omega \big[\Delta^2 y - (1-\mu)\delta(k\,dy)\big] H(y)\, dx \\
& - \frac{1}{2} \int_\Omega (\operatorname{div} H - 1) |y'|^2 \, dx - C_0 \int_\Gamma \left\{ v_2 y - v_1 \frac{\partial y}{\partial \nu} \right\} d\Gamma \\
& + \frac{1}{2} \big\{ (\gamma C_0)^2 + R \big\} \int_\Gamma \left\{ |y'|^2 + \left| \frac{\partial y'}{\partial \nu} \right|^2 \right\} d\Gamma.
\end{aligned} \tag{3.59}$$

Since $v_1, v_2 \in L^2(\Gamma)$, we apply Lemma 3.3 to (3.59), then

$$\begin{aligned}
p'(t) \leqslant{} & -\frac{1}{2} \int_\Omega a(y,y)\, dx - \frac{1}{2} \int_\Omega (\operatorname{div} H - 1) |y'|^2 \, dx \\
& + C\theta_1 \int_\Gamma |v_1|^2 \, d\Gamma + C\theta_2 \int_\Gamma |v_2|^2 \, d\Gamma + C \int_\Gamma \left(\alpha |y|^2 + \beta \left| \frac{\partial y}{\partial \nu} \right|^2 \right) d\Gamma \\
& - C_0 \int_\Gamma \left\{ v_2 y - v_1 \frac{\partial y}{\partial \nu} \right\} d\Gamma + \frac{1}{2} \big\{ (\gamma C_0)^2 + R \big\} \int_\Gamma \left\{ |y'|^2 + \left| \frac{\partial y'}{\partial \nu} \right|^2 \right\} d\Gamma \\
& + \operatorname{lot}(y),
\end{aligned} \tag{3.60}$$

where $\theta_1 \geqslant 1$, $\theta_2 \geqslant 1$ are the constants to be determined later.

On the other hand, by (3.17)–(3.18), we have

$$v_1^2 \leqslant -2\beta_1 v_1 \frac{\partial y}{\partial \nu} - \beta_1 \beta \left|\frac{\partial y}{\partial \nu}\right|^2 + \frac{\beta_1}{\beta_0} h^2\left(\frac{\partial y'}{\partial \nu}\right), \tag{3.61}$$

$$v_2^2 \leqslant 2\alpha_1 v_2 y - \alpha_1 \alpha |y|^2 + \frac{\alpha_1}{\alpha_0} g^2(y'). \tag{3.62}$$

It follows that

$$\begin{aligned} p'(t) \leqslant{} & -\frac{1}{2}\int_\Omega a(y,y)\,dx - \frac{1}{2}\int_\Omega (\operatorname{div} H - 1)|y'|^2\,dx + (C_0 - 2\beta_1 C\theta_1)\int_\Gamma v_1 \frac{\partial y}{\partial \nu}\,d\Gamma \\ & + (2\alpha_1\theta_2 C - C_0)\int_\Gamma v_2 y\,d\Gamma + (C - \alpha_1\theta_2 C)\int_\Gamma \alpha|y|^2\,d\Gamma \\ & + (C\beta - \theta_1 C\beta_1)\int_\Gamma \beta\left|\frac{\partial y}{\partial \nu}\right|^2 d\Gamma + \frac{\alpha_1\theta_2 C}{\alpha_0}\int_\Gamma g^2(y')\,d\Gamma + \frac{\beta_1\theta_1 C}{\beta_0}\int_\Gamma h^2\left(\frac{\partial y'}{\partial \nu}\right) d\Gamma \\ & + \frac{1}{2}\left\{(\gamma C_0)^2 + R\right\}\int_\Gamma \left\{|y'|^2 + \left|\frac{\partial y'}{\partial \nu}\right|^2\right\} d\Gamma + \operatorname{lot}(y). \end{aligned} \tag{3.63}$$

Taking $C_0 - 2\beta_1\theta_1 C = 0$, $C_0 - 2\alpha_1\theta_2 C = 0$, we get $\theta_1 = \theta_2 \frac{\alpha_1}{\beta_1}$. Therefore

$$\begin{aligned} p'(t) \leqslant{} & -E(t) + (C - \alpha_1\theta_2 C + 1/2)\int_\Gamma \alpha|y|^2\,d\Gamma \\ & + (C\beta - C\theta_1\beta_1 + 1/2)\int_\Gamma \beta\left|\frac{\partial y}{\partial \nu}\right|^2 d\Gamma \\ & + \frac{1}{2}\left\{(\gamma C_0)^2 + R\right\}\int_\Gamma \left\{|y'|^2 + \left|\frac{\partial y'}{\partial \nu}\right|^2\right\} d\Gamma \\ & + \alpha_1\theta_2 C\left\{\frac{1}{\alpha_0}\int_\Gamma g^2(y')\,d\Gamma + \frac{1}{\beta_0}\int_\Gamma h^2\left(\frac{\partial y'}{\partial \nu}\right) d\Gamma\right\} + \operatorname{lot}(y). \end{aligned} \tag{3.64}$$

Now, we take $\theta_2 \geqslant \max\{1, \frac{C+1/2}{\alpha_1 C}, \frac{C\beta+1/2}{\alpha_1 C}\}$, then $C - \alpha_1\theta_2 C + 1/2 \leqslant 0$, $C\beta - \theta_1 C\beta_1 + 1/2 \leqslant 0$, $\theta_1 \geqslant 1$ and $\theta_2 \geqslant 1$. Thus, we obtain

$$p'(t) \leqslant -E(t) + C_2\int_\Gamma \left(|y'|^2 + \left|\frac{\partial y'}{\partial \nu}\right|^2\right) d\Gamma + C_3\int_\Gamma \left[g^2(y') + h^2\left(\frac{\partial y'}{\partial \nu}\right)\right] d\Gamma + \operatorname{lot}(y) \tag{3.65}$$

with some constants C_2, C_3. Finally, by a compactness (uniqueness) arguments as in [18], the lower order terms in (3.65) can be absorbed. We complete the proof of Lemma 3.5. □

4. Proof of the main results

Proof of Theorem 3.1. It should be mentioned that our proof here is similar to that in [1]. For the sake of completeness, we give the sketch of the proof.

For $\forall\epsilon > 0$, we define the perturbed energy by

$$E_\epsilon(t) = E(t) + \epsilon p(t)\big(E(t)\big)^{(p-1)/2}. \tag{4.1}$$

Since the energy $E(t)$ is nonincreasing in t, for any given $M > 1$, a straight computation shows that

$$M^{-1/2}\big(E_\epsilon(t)\big)^{(p+1)/2} \leqslant \big(E(t)\big)^{(p+1)/2} \leqslant M^{1/2}\big(E_\epsilon(t)\big)^{(p+1)/2}, \tag{4.2}$$

provided ϵ is small enough such that

$$\epsilon \leqslant \big(E(0)\big)^{(1-p)/2}\big(1 - M^{-1/(p+1)}\big)/C_1.$$

Then

$$E'_\epsilon(t) = E'(t) + \epsilon\frac{p-1}{2}\big(E(t)\big)^{(p-3)/2}p(t)E'(t) + \epsilon\big(E(t)\big)^{(p-1)/2}p'(t). \tag{4.3}$$

Using (3.21), (3.23) and (3.24), we deduce from (3.54) that

$$p'(t) \leqslant -\gamma_0 E(t) + C_4\int_\Gamma\left\{|y'|^2 + \left|\frac{\partial y'}{\partial\nu}\right|^2\right\}d\Gamma. \tag{4.4}$$

Inserting (3.13), (3.53) and (4.4) into (4.3), we obtain

$$\begin{aligned} E'_\epsilon(t) \leqslant{}& \left(-1 - \epsilon\frac{p-1}{2}C_1\big(E(0)\big)^{(p-1)/2}\right)\int_\Gamma\left(g(y')y' + h\left(\frac{\partial y'}{\partial\nu}\right)\left(\frac{\partial y'}{\partial\nu}\right)\right)d\Gamma \\ &+ \epsilon C_4\big(E(t)\big)^{(p-1)/2}\int_\Gamma\left(|y'|^2 + \left|\frac{\partial y'}{\partial\nu}\right|^2\right)d\Gamma - \epsilon\gamma_0\big(E(t)\big)^{(1+p)/2}. \end{aligned} \tag{4.5}$$

(i) If $p = 1$, then from (3.17)–(3.20), (3.23)–(3.24) we have

$$E'_\epsilon(t) \leqslant \big(-1 + \epsilon C_4(L_1 + L_3)\big)\int_\Gamma\left(g(y')y' + h\left(\frac{\partial y'}{\partial\nu}\right)\left(\frac{\partial y'}{\partial\nu}\right)\right) - \epsilon\gamma_0\big(E(t)\big). \tag{4.6}$$

By choosing $\epsilon C_4(L_1 + L_3) \leqslant 1$, we obtain

$$E'_\epsilon(t) \leqslant -\epsilon\gamma_0 E(t) \leqslant -\epsilon\gamma_0 M^{-1/2}E_\epsilon(t), \quad \forall t \geqslant 0. \tag{4.7}$$

Then (4.7) and (4.2) imply that

$$E(t) \leqslant ME(0)\exp\big(-\epsilon\gamma_0 M^{-1/2}\big) = ME(0)\exp(-\omega t), \quad \forall t \geqslant 0,$$

with the constant $\omega = \epsilon\gamma_0 M^{-1/2}$.

Notice that the constant ϵ does not depend on $E(0)$, and hence the constant ω defined above does not depend on $E(0)$ either.

(ii) If $p > 1$, then from (3.17)–(3.20) and (3.23)–(3.24), we have

$$|s|^{p+1} \leqslant L_1 g(s)s, \quad \forall|s| \leqslant 1; \qquad |s|^2 \leqslant L_1 g(s)s, \quad \forall|s| \geqslant 1; \tag{4.8}$$

$$|s|^{p+1} \leqslant L_3 h(s)s, \quad \forall|s| \leqslant 1; \qquad |s|^2 \leqslant L_3 h(s)s, \quad \forall|s| \leqslant 1. \tag{4.9}$$

Using (4.8), we have

$$\epsilon C_4\big(E(t)\big)^{(1-p)/2}\int_\Gamma |y'|^2\,d\Gamma$$
$$=\epsilon C_4\big(E(t)\big)^{(p-1)/2}\int_{\{|y'|\geqslant 1\}} |y'|^2\,d\Gamma+\epsilon C_4\big(E(t)\big)^{(p-1)/2}\int_{\{|y'|\leqslant 1\}} |y'|^2\,d\Gamma$$
$$\leqslant \epsilon C_4 L_1\big(E(0)\big)^{(p-1)/2}\int_{\{|y'|\geqslant 1\}} g(y')y'\,d\Gamma+\epsilon C_4\big(E(t)\big)^{(p-1)/2}\int_{\{|y'|\leqslant 1\}} |y'|^2\,d\Gamma, \tag{4.10}$$

with the exponents $\alpha_1=(p+1)/(p-1)$, $\alpha_2=(p+1)/2$, applying Young's inequality to the second term in (4.10), we obtain

$$\epsilon C_4\big(E(t)\big)^{(p-1)/2}\int_{\{|y'|\leqslant 1\}} |y'|^2\,d\Gamma$$
$$\leqslant \frac{\epsilon\gamma_0}{4}\big(E(t)\big)^{(p+1)/2}+\epsilon\big(4\gamma_0^{-1}C_4\big)^{(p+1)/2}\bigg(\int_{\{|y'|\leqslant 1\}} |y'|^2\,d\Gamma\bigg)^{(p+1)/2}. \tag{4.11}$$

In turn with the exponents $\alpha_1=(p+1)/(p-1)$, $\alpha_2=(p+1)/2$, applying Hölder's inequality to the integral term $(\int_{\{|y'|\leqslant 1\}}|y'|^2\,d\Gamma)^{(p+1)/2}$ together with (4.8), we obtain

$$\bigg(\int_{\{|y'|\leqslant 1\}} |y'|^2\,d\Gamma\bigg)^{(p+1)/2}\leqslant\bigg(\int_{\{|y'|\leqslant 1\}} d\Gamma\bigg)^{(p-1)/2}\int_{\{|y'|\leqslant 1\}} |y'|^{p+1}\,d\Gamma$$
$$\leqslant (\operatorname{meas}\Gamma)^{(p-1)/2}L_1\int_{\{|y'|\leqslant 1\}} g(y')y'\,d\Gamma. \tag{4.12}$$

Inserting (4.11) and (4.12) into (4.10) gives

$$\epsilon C_4\big(E(t)\big)^{(p-1)/2}\int_\Gamma |y'|^2\,d\Gamma$$
$$\leqslant \frac{\epsilon\gamma_0}{4}\big(E(t)\big)^{(p+1)/2}+\epsilon L_1\big\{C_4\big(E(0)\big)^{(p-1)/2}$$
$$+(\operatorname{meas}\Gamma)^{(p-1)/2}\big(4\gamma_0^{-1}C_4\big)^{(p-1)/2}\big\}\int_\Gamma g(y')y'\,d\Gamma. \tag{4.13}$$

Similarly, using (4.9), we show that

$$\epsilon C_4\big(E(t)\big)^{(p-1)/2}\int_\Gamma \left|\frac{\partial y'}{\partial\nu}\right|^2 d\Gamma$$
$$\leqslant \frac{\epsilon\gamma_0}{4}\big(E(t)\big)^{(p+1)/2}+\epsilon L_3\big\{C_4\big(E(0)\big)^{(p-1)/2}$$
$$+(\operatorname{meas}\Gamma)^{(p-1)/2}\big(4\gamma_0^{-1}C_4\big)^{(p-1)/2}\big\}\int_\Gamma h\bigg(\frac{\partial y'}{\partial\nu}\bigg)\frac{\partial y'}{\partial\nu}\,d\Gamma. \tag{4.14}$$

Plugging (4.13)–(4.14) into (4.5), it follows

$$E'_\epsilon(t) \leqslant -\frac{\epsilon\gamma_0}{2}\big(E(t)\big)^{(p+1)/2} \leqslant -\frac{\epsilon\gamma_0}{2}M^{-1/2}\big(E_\epsilon(t)\big)^{(p+1)/2}, \tag{4.15}$$

provided ϵ is chosen such that

$$\epsilon\bigg\{(L_1+L_3)\bigg(C_4\big(E(0)\big)^{(p-1)/2}+\frac{p-1}{2}C_1\big(E(0)\big)^{(p-1)/2} + (\text{meas}\,\Gamma)^{(p-1)/2}\big(4\gamma_0^{-1}C_4\big)^{(p-1)/2}\bigg)\bigg\} \leqslant 1.$$

Combining (4.5) with (4.2), we obtain

$$\begin{aligned} E(t) &\leqslant ME(0)\bigg\{1+\epsilon\frac{p-1}{4}M^{-p/(p+1)}\big(E(0)\big)^{(p-1)/2}\bigg\}^{-2/(p-1)} \\ &= ME(0)(1+\omega t)^{-2/(p-1)}, \quad \forall t \geqslant 0, \end{aligned} \tag{4.16}$$

with the constant $\omega = \epsilon\gamma_0\frac{p-1}{4}M^{-p/(p+1)}(E(0))^{(p-1)/2}$. The proof of Theorem 3.1 is thus complete. $\square$

Proof of Theorem 3.2. First, using (3.26)–(3.27), it follows from (3.54) that

$$p'(t) \leqslant -\gamma_0 E(t) + C_6\int_\Gamma \bigg(\big|g(y')\big|^2 + \bigg|h\bigg(\frac{\partial y'}{\partial\nu}\bigg)\bigg|^2\bigg)\,d\Gamma, \tag{4.17}$$

with the constant $C_6 = C_3 + C_2(L_1^2 + L_3^2)$.

Now we introduce the perturbed energy

$$E_\epsilon(t) = E(t) + \epsilon p(t)\big(E(t)\big)^{(1-p)/2p}. \tag{4.18}$$

Then, for any $M > 1$, we have

$$M^{-1/2}\big(E_\epsilon(t)\big)^{(p+1)/2p} \leqslant \big(E(t)\big)^{(p+1)/2p} \leqslant M^{1/2}\big(E_\epsilon(t)\big)^{(p+1)/2p}, \tag{4.19}$$

provided ϵ small enough such that

$$\epsilon \leqslant \big(E(0)\big)^{(p-1)/2p}\big(1-M^{-p/(p+1)}\big)/C_1. \tag{4.20}$$

A simple computation shows

$$E'_\epsilon(t) = E'(t) + \epsilon\frac{1-p}{2p}\big(E(t)\big)^{(1-3p)/2p}p(t)E'(t) + \epsilon\big(E(t)\big)^{(p-1)/2p}p'(t). \tag{4.21}$$

Substituting (3.13), (3.53) and (4.7) into (4.21) gives

$$\begin{aligned} E'_\epsilon(t) \leqslant{}& \bigg(-1+\epsilon\frac{1-p}{2p}C_1\big(E(0)\big)^{(1-p)/2p}\bigg)\int_\Gamma\bigg(g(y')y' + h\bigg(\frac{\partial y'}{\partial\nu}\bigg)\frac{\partial y'}{\partial\nu}\bigg)\,d\Gamma \\ &+ \epsilon C_6\big(E(t)\big)^{(1-p)/2p}\int_\Gamma\bigg(g^2(y') + h^2\bigg(\frac{\partial y'}{\partial\nu}\bigg)\bigg)\,d\Gamma - \epsilon\gamma - 0\big(E(t)\big)^{(1+p)/2p}. \end{aligned} \tag{4.22}$$

On the other hand, by the conditions (3.26)–(3.27), we have

$$\left|g(s)\right|^2 \leqslant L_0^p g(s)s, \quad \text{for } |s| \geqslant 1; \qquad \left|g(s)\right|^{(p+1)/p} \leqslant L_0 g(s)s, \quad \text{for } |s| \leqslant 1; \tag{4.23}$$

$$\left|h(s)\right|^2 \leqslant L_2^p h(s)s, \quad \text{for } |s| \geqslant 1; \qquad \left|h(s)\right|^{(p+1)/p} \leqslant L_2 h(s)s, \quad \text{for } |s| \leqslant 1. \tag{4.24}$$

We rewrite

$$\begin{aligned}\epsilon C_6\big(E(t)\big)^{(1-p)/2p}\int_\Gamma g^2(y')\,d\Gamma &= \epsilon C_6\big(E(t)\big)^{(1-p)/2p}\int_{\{|y'|\geqslant 1\}} g^2(y')\,d\Gamma\\ &\quad + \epsilon C_6\big(E(t)\big)^{(1-p)/2p}\int_{\{|y'|\leqslant 1\}} g^2(y')\,d\Gamma.\end{aligned} \tag{4.25}$$

Using (4.23), the first term in the right-hand side of (4.25) can be controlled by

$$\epsilon C_6 L_0^p\big(E(0)\big)^{(1-p)/2p}\int_{\{|y'|\geqslant 1\}} g(y')y'\,d\Gamma,$$

with the exponents $\alpha_1 = (1+p)/(1-p), \alpha_2 = (1+p)/2p$, applying Young's inequality to the second term in (4.25), we obtain

$$\begin{aligned}&\epsilon C_6\big(E(t)\big)^{(1-p)/2p}\int_{\{|y'|\leqslant 1\}} g^2(y')\,d\Gamma\\ &\quad\leqslant \frac{\epsilon\gamma_0}{4}\big(E(t)\big)^{(1+p)/2p} + \epsilon\big(4\gamma_0^{-1}C_6\big)^{(1+p)/2p}\bigg(\int_{\{|y'|\leqslant 1\}} g^2(y')\,d\Gamma\bigg)^{(1+p)/2p}.\end{aligned} \tag{4.26}$$

In turn with the exponents $\alpha_1 = (1+p)/(1-p), \alpha_2 = (1+p)/2p$, applying Hölder's inequality to the integral term $(\int_{\{|y'|\leqslant 1\}} g^2(y')\,d\Gamma)^{(1+p)/2p}$, together with (4.23), we obtain

$$\begin{aligned}&\epsilon\big(4\gamma_0^{-1}C_6\big)^{(1-p)/2p}\bigg(\int_{|y'|\leqslant 1} g^2(y')\,d\Gamma\bigg)^{(1+p)/2p}\\ &\quad\leqslant \epsilon\big(4\gamma_0^{-1}C_6\big)^{(1-p)/2p}\bigg(\int_{|y'|\leqslant 1} d\Gamma\bigg)^{(1-p)/2p}\bigg(\int_{|y'|\leqslant 1}\left|g(y')\right|^{(1+p)/p}\bigg)\\ &\quad\leqslant \epsilon\big(4\gamma_0^{-1}C_6\big)^{(1-p)/2p} L_0(\operatorname{meas}\Gamma)^{(1-p)/2p}\int_{|y'|\leqslant 1} g(y')y'\,d\Gamma.\end{aligned} \tag{4.27}$$

Inserting (4.26)–(4.27) into (4.25) gives

$$\begin{aligned}&\epsilon C_6\big(E(t)\big)^{(1-p)/2p}\int_\Gamma g^2(y')\,d\Gamma\\ &\quad\leqslant \frac{\epsilon\gamma_0}{4}\big(E(t)\big)^{(1+p)/2p}\\ &\qquad + \epsilon\big\{C_6L_0^p\big(E(0)\big)^{(1-p)/2p} + \big(4\gamma_0^{-1}C_6\big)^{(1-p)/2p}L_0(\operatorname{meas}\Gamma)^{(1-p)/2p}\big\}\int_\Gamma g(y')y'\,d\Gamma.\end{aligned} \tag{4.28}$$

Similarly, using (4.24), we deduce that

$$\begin{aligned}
&\epsilon C_6\big(E(t)\big)^{(1-p)/2p}\int\limits_\Gamma h^2\left(\frac{\partial y'}{\partial\nu}\right)d\Gamma\\
&\quad\leqslant \frac{\epsilon\gamma_0}{4}\big(E(t)\big)^{(1+p)/2p}\\
&\qquad+\epsilon\big\{C_6L_2^p\big(E(0)\big)^{(1-p)/2p}\\
&\qquad+\big(4\gamma_0^{-1}C_6\big)^{(1-p)/2p}L_2(\operatorname{meas}\Gamma)^{(1-p)/2p}\big\}\int\limits_\Gamma h\left(\frac{\partial y'}{\partial\nu}\right)\left(\frac{\partial y'}{\partial\nu}\right)d\Gamma. \qquad (4.29)
\end{aligned}$$

Inserting (4.28) and (4.29) into (4.22), together with (4.19), it gives

$$E'_\epsilon(t)\leqslant -\frac{1}{2}\epsilon\gamma_0\big(E(t)\big)^{(p+1)/2p}\leqslant -\frac{1}{2}\epsilon\gamma_0M^{-1/2}\big(E_\epsilon(t)\big)^{(p+1)/2p}, \qquad (4.30)$$

provided ϵ is chosen such that

$$\begin{aligned}
&\epsilon\left\{\left(C_6\big(L_0^p+L_2^p\big)+\frac{1-p}{2p}C_1\right)\big(E(0)\big)^{(p+1)/2p}\right.\\
&\quad\left.+(L_0+L_2)(\operatorname{meas}\Gamma)^{(1-p)/2p}\big(4\gamma_0^{-1}C_6\big)^{(1-p)/2p}\right\}\leqslant 1.
\end{aligned}$$

Combining (4.30) and (4.19), we obtain

$$\begin{aligned}
E(t)&\leqslant ME(0)\left\{1+\epsilon\frac{1-p}{4p}\gamma_0M^{-1/(p+1)}\big(E(0)\big)^{(1-p)/2p}\right\}^{-2p/(1-p)}\\
&=M\big(E(0)\big)(1+\omega t)^{-2p/(1-p)},\quad \forall t\geqslant 0, \qquad (4.31)
\end{aligned}$$

with the constant $\omega=\epsilon\frac{1-p}{4p}\gamma_0M^{-1/(p+1)}(E(0))^{(1-p)/2p}$. The proof of Theorem 3.2 is then complete. □

Remark 4.1. Boundary feedback control acts through a nonlinear feedback law defined in terms of both the position and the velocity. Typical feedback involves only the velocity, while the position is included to guarantee uniqueness of the solution when control is acting on the entire boundary. Velocity feed back alone suffices if control acts through a portion of the boundary while the remaining portion is clamped. So, writing $\Gamma_0=\{x\in\Gamma\mid H.\nu\geqslant 0\}$ and $\Gamma_1=\Gamma-\Gamma_0$, we can replace the boundary condition in (3.1) by

$$\begin{cases}
\Delta y+(1-\mu)B_1y=-\beta\frac{\partial y}{\partial\nu}-h\big(\frac{\partial y'}{\partial\nu}\big) & \text{on } \Gamma_0\times[0,\infty),\\
\frac{\partial\Delta y}{\partial\nu}+(1-\mu)B_2y=\alpha y+g(y') & \text{on } \Gamma_0\times[0,\infty),\\
y=\frac{\partial y}{\partial\nu}=0 & \text{on } \Gamma_1\times[0,\infty).
\end{cases}$$

If we assume that $\bar\Gamma_0\cap\bar\Gamma_1=\emptyset$, we can prove the same results as those in Theorems 3.1 and 3.2 for any $\alpha\geqslant\alpha_0>0$ and $\beta\geqslant\beta_0>0$. Note that without the geometrical condition $\bar\Gamma_0\cap\bar\Gamma_1=\emptyset$, in the case $\alpha=\beta=0$ and $\Gamma_1\neq\emptyset$, the estimate of the rates of decay of the energy have been obtained by Lagnese [13] for the constant coefficients case.

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